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A codicity undecidable problem in the plane

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Abstract

In this paper we give a new undecidability result about tiling problems. Given a finite set of polyomino types, the problem whether this set is a *code*, is undecidable. The same result holds for dominoes.

Keywords: Polyomino; Tiling; Undecidability; Code; Domino tiling

1. Introduction

Planar tiling problems have been widely studied in the literature [1,2,6–9]. Concerning the problem of the existence of a plane tiling, two versions exist, in the first one, puzzle pieces are squares with outer and inner hooks along the edges which force some pairs of pieces to be adjacent to each other. In the second version, pieces are squares having colors along their edges and squares can be adjacent only if the pair of colors of their common edge belong to some given set (usually colors are needed to be equal, and these squares are called *dominoes*). Concerning the classical tiling problem i.e. is it possible to tile the plane with translated copies of a finite set of tiles, the two versions have been proved to be equivalent and the problem is undecidable [3,9]. The proof is based on a simulation of a Turing Machine. The problem we deal with in this paper concerns the property of codicity.

Let $\mathscr{C}$ be a finite set of polyominoes [6] (a polyomino is a connected finite set of unit squares without holes). Is $\mathscr{C}$ a code, in other words, does every polyomino tilable with translated copies of elements of $\mathscr{C}$ be tilable in a unique way?

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We solve this problem by encoding the Post problem [5]. It does not seem easy to encode the behavior of a Turing machine. It is not actually surprising because the nature of the problem to be solved, that is the existence of *two* solutions for some polyomino, looks more like the Post problem which consists also in providing two writings of some word. We encode the Post problem using squares with outer and inner hooks along the edges and we call them “colored squares”. This encoding permits to derive easily the undecidability of the codicity problem for dominoes.

Section 2 establishes our notations. In Section 3 we build squares with inner and outer hooks on their edges and prove some properties of tilings with these squares. The last section presents the proofs of the two undecidability problems we consider, namely for polyominoes and for dominoes.

2. Preliminaries

Let E be a plane, with origin O. A *cell* is a unit square, whose vertices have integer coordinates. A *polyomino* is a finite union of nonoverlapping cells, whose outline is a simple closed curve. Thus, a polyomino is a connected set without holes. Two polyominoes are *equivalent* if one of them is the translated image of the other one. The equivalence class of a polyomino p is a *polyomino type* P and p is an *instance* of P. Every polyomino type P has a particular element, called its canonical one, namely the polyomino whose leftmost vertex among the highest ones is located in the origin of the plane. Henceforth, a polyomino type P is identified with its canonical element, and with any of its elements if no ambiguity arises.

It is convenient to consider a polyomino as a subset of $\mathbf{Z}^2$, identifying each cell of p with its lowest left corner. Let $\mathscr{P}=(P_i)_{i\in I}$ a family of polyomino's types, and let $p\subset\mathbf{Z}^2$ be a polyomino.

A *tiling of* p *with* $\mathscr{P}$ is a collection of polyominoes $p_n\subset\mathbf{Z}^2$ such that p is the disjoint union of the p_n, and each p_n's type is an element of $\mathscr{P}$. A polyomino is said to be *tilable with* $\mathscr{P}$ if there exists a tiling of p with $\mathscr{P}$.

Let $u\in\mathbf{Z}^2$, and p be a polyomino or a point. We denote by p_u the translated image of p by u. One can observe that the property for a polyomino to be tilable with $\mathscr{P}$ is invariant by translation.

Fig. 1 represents two tilings of a square of size 2 with two bars (a vertical and a horizontal one).

Let $\mathscr{P}$ be a *finite* set of polyomino's types. The set $\mathscr{P}$ is a *code* if each polyomino tilable with $\mathscr{P}$ admits a unique tiling with $\mathscr{P}$.

For example the set $\{b_v,b_h\}$ is not a code.

Our purpose is to prove that the following problem is undecidable:

$$\textit{Let } \mathscr{P} \textit{ be a finite set of polyomino's types, is } \mathscr{P} \textit{ a code?}$$

We solve this question in a classical way, encoding the Post problem by means of *colored squares* defined in the next section.

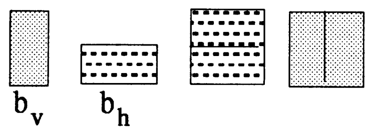

Fig. 1. $\mathscr{P}=\{b_v,b_h\}$, two tilings of a square with $\mathscr{P}$.

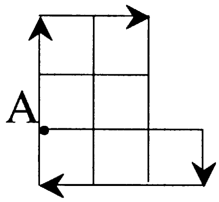

Fig. 2. The contour word of p with origin A is $uurrddrdlllu$.

3. Colored squares

Using the classical encoding of a picture due to Freeman [4], we consider the four letter alphabet $X=\{u,d,l,r\}$ (for up, down, left right). The boundary $b(p)$ is clockwise oriented. The *length* of the boundary is denoted by $|b(p)|$. With every point of $b(p)$ with integer coordinates, there is associated a word m on the alphabet X, which describes the sequence of moves along the boundary, starting from A. The word m will be called the *contour word with origin A*. When the starting point is changed, the obtained word is a conjugate of m. Clearly, the length $|m|$ of the word m is equal to $|b(p)|$ (Fig. 2).

Let N be an integer ($N>5$). Let us consider a square C of size $2N+1$. We will construct a family of polyominoes obtained from C, by putting on the north and west sides of C outer "hooks" and on the south and east sides inner "hooks" with variable size. The obtained polyomino will be called a *colored square*, each side having a color corresponding to the size of the "hook".

More precisely, a north side has color i ($0<i<N-4$, outer hook) (resp. color 0) if this side is encoded with the word (Fig. 3):

$$r^N u^i lur^3 dld^i r^N, \tag{1}$$

resp. with the word:

$$r^{2N+1}. \tag{2}$$

For a west side, the definition is the corresponding one (outer hook), the word encoding the side is obtained by the circular permutation $r\to u\to l\to d\to r$.

An east side has color $-i$ ($0<i<N-4$) (resp. color 0) if it is encoded with the word obtained from the word (3) (resp. (3)) interchanging letters r and d, and letters u and l (Fig. 4).

Finally, a south side has color $-i$ (resp. color 0) if it is encoded with the word obtained from the word (3) (resp. (3)) interchanging letters r and l. In order to obtain disjoint inner hooks on the east and south sides we need to suppose $N>5$.

The colored square with colors i_1,i_2,i_3,i_4, respectively on east, south, west and north sides will be denoted by $c(i_1,i_2,i_3,i_4)$ and will be represented as in Fig. 5.

We denote by $\mathscr{C}_N$ the set of colored squares of size $2N+1$ for $N>5$.

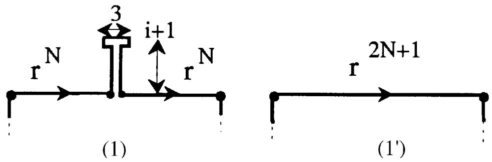

Fig. 3. Color of a north side.

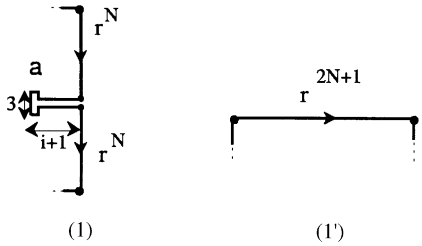

Fig. 4. Color of an east side.

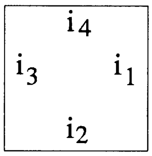

Fig. 5. A colored square.

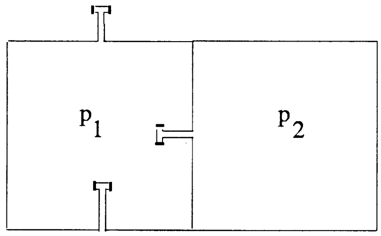

Fig. 6. Square p_2 is an east neighbor of square p_1.

Let c be a colored square and $x \in \{E, S, W, N\}$ be a direction. The color of square c on the side x will be denoted by $col_x(c)$.

Let p_1 and p_2 two colored squares. These squares are *neighbor squares* if their centers are on the same horizontal or vertical line, the distance between them is equal to $2N+1$ and the colors of the squares along their common side are opposite colors. In other words, the hooks on the two squares fit into each other (an outer hook for one square, an inner hook for the other one) (Fig. 6).

The square p_2 with center a_1 is an *east neighbor* of p_1 with center a_2, iff p_1 and p_2 are neighbor and they fit along the east side of p_1 more precisely iff:

$$a_2 = a_{1_{(2N+1,0)}} \quad \text{and} \quad col_{E(p_1)} = -col_{W(p_2)}.$$

North, *west*, *south neighbors* are defined in a similar way.

3.1. Properties of colored squares

The special shape of the hooks implies (they have been built for this purpose) several invariance properties for the tilings of polyominoes with colored squares.

Let T be a set of colored squares. The colored square of T whose center is the leftmost among the highest ones is called the *N–W maximal square*.

Lemma 1. *Let p be a polyomino tiled with colored squares in $\mathscr{C}_N$. Then, the set of centers of the colored squares of a tiling T of p does not depend on T but only on p.*

Proof. Let T and T' be two different tilings of p and assume that their associated sets of centers $C(T)$ and $C(T')$ are different. Let x be the center of a N–W maximal square c in the symmetric difference of $C(T)$ and $C(T')$ and let us assume that $x \notin C(T')$ and $x \in C(T)$. It turns out that the $N \times 1$ horizontal bar located in the N–W corner of c is included in p (Fig. 7). Thus, it has to be covered by elements of T'. In the same way, the $1 \times N$ vertical bar located in the N–W corner of c is included in p. So the union of these two bars is covered by elements of T'. Considering the shape of the hooks, it is impossible that each bar can be covered in T' only with cells belonging to outer hooks. So we can suppose that there exist two different cells a and b belonging respectively to the horizontal and the vertical bar such that a and b do not belong to any hook in T'. Let x_a (resp. x_b) be the center of the square in T' covering a (resp. b). Then, x_a and x_b belong respectively to the underlying squares of a and b, (the underlying square of a colored square c is a square of size $2N+1$ and with the same center as c.

However, in a tiling, two underlying squares of colored squares cannot have any common cell. Indeed, two colored squares whose underlying squares are not disjoint cannot be disjoint. Therefore, x_a is not center of an element of T, otherwise the colored squares of T with center x and x_a would have nondisjoint underlying squares. In the same way, x_b is not the center of an element of T. Thus, x_a and x_B belong to the symmetric difference of $C(T)$ and $C(T')$, hence, according to the fact that x is N–W maximal, x_a has to be located on the half-right horizontal straight line d with origin x (in order to cover a, and x_b has to be located on the half-down vertical straight line d' with origin x (in order to cover b). Let e be the N–W corner cell of c. The distance between x and x_a is less than or equal to the distance between e and a, otherwise the square of T' with center x_a could not cover a. For the same reason, the distance between x and x_b is less than or equal to the distance between e and b. So, the distance between x_a and x_b is less than or equal to the distance between the centers of a and b. But $x \neq x_a$ and $x \neq x_b$ so $x_a \neq x_b$. This implies that the two squares of size $2N+1$ with center x_a and x_b are overlapping, and this cannot happen since x_a and x_b are centers of colored squares belonging to a same tiling, there is a contradiction. □

Lemma 2. *Let p be a polyomino and a be a cell of p. If there exists a tiling T of p with $\mathscr{C}_N$ such that a is the central cell of a colored square c in this tiling*, *and if c has no east* (*resp. south*, *west*, *north*) *neighbor in T then the east* (*resp. south*, *west*, *north*) *color of c does not depend on T but only on p and is computable from p and a.*

Proof. Let us suppose that c does not have an east neighbor in T (Figs. 8).

The value of $col_E(c)$ can be computed from p and a. Actually, $col_E(c) \neq 0$ iff the boundary of the inner hook of c (the bold part Fig. 8) belongs to the boundary of p, and it can be computed from p and a. □

A property of a polyomino p tilable with $\mathscr{C}_N$, which does not depend on the tiling of p will be called an *invariant of p*. Let p be a polyomino, T be a tiling of p with $\mathscr{C}_N$, and S be the set of centers of the elements of T. For a center $a \in S$ we denote by $c_T(a)$ the colored square of T whose center is a. We define a graph $G=(S,F)$ as

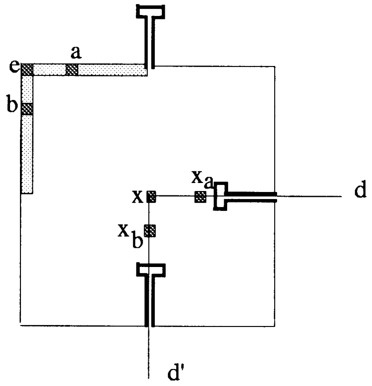

Fig. 7.

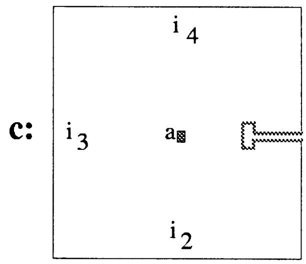

Fig. 8.

follows:

$$(a,b) \in F \quad \text{if and only if } c_T(b) \text{ is an east or south neighbor of } c_T(a).$$

Lemma 3. *The graph $G=(S,F)$ is an invariant of p.*

Proof. We use Lemma 1. Let c be a colored square in a tiling T of p. We know that the center of this square depends only on p and not on T. Therefore, if $b=a_{(2N+1,0)}$ then (a,b) is an edge of G iff the cells $a_{(N,0)}$ and $a_{(N+1,0)}$ belong to p, and if $b=a_{(0,-2N-1)}$ then (a,b) is an edge of G iff the cells $a_{(-N,0)}$ and $a_{(-N-1,0)}$ belong to p. Thus, we have proved that G depends only on p. □

4. Main result

In this section we prove the main result of this paper, namely, the undecidability for a finite set of polyominoes to be a code. The proof is obtained by encoding the Post problem. We recall its statement:

Let $A=\{a_1,\dots,a_p\}$ be a finite alphabet, X, Y be two finite sequences $X=(x_1,\dots,x_k)$, $Y=(y_1,\dots,y_k)$ of nonempty words on the alphabet A such that for every $i\in\{1,\dots,k\}$, $x_i \neq y_i$.

One cannot decide whether there exists a finite sequence of integers $i_1,\dots,i_n\in\{1,\dots,k\}$, $n\geqslant 2$, such that $x_{i_1}\dots x_{i_n}=y_{i_1}\dots y_{i_n}$.

We describe now a set of polyominoes $\mathscr{C}$ such that the Post problem has a solution if and only if $\mathscr{C}$ is not a code.

Let $\mathscr{M}$ be the following set of symbols:

$$\mathscr{M} = \bigcup_{i\in\{1,\dots,k\}} \{x_i, y_i, e_{x_i}, e_{y_i}, I_i\} \cup \{a_j \mid j\in\{1,\dots,p\}\}$$
$$\cup\, \{x,y\} \cup \{x',y'\} \cup \{b_x, b_y\}$$

$|M| = 5k+p+6 = m_0$.

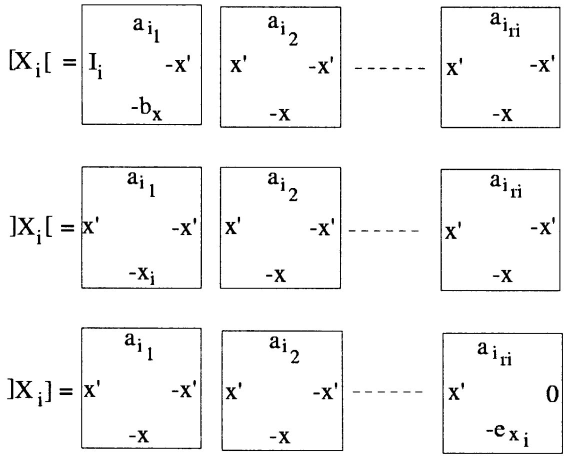

Fig. 9.

We define a one to one mapping from $\mathscr{M}$ onto $\{1,\ldots,m_0\}$, so from now, each element of $\mathscr{M}$ is confused with its image in $\{1,\ldots,m_0\}$.

Let m be an integer such that $m > m_0$. The set $\mathscr{C}$ is a set of polyominoes obtained by matching several colored squares of $\mathscr{C}_m$ in a horizontal line:

for every word x_i, $i \in \{1,\ldots,k\}$, we define three polyominoes denoted respectively by $[X_i[,]X_i[,]X_i]$:

Let $x_i = a_{i_1}\ldots a_{i_{r_i}}$. To x_i we associate three horizontal packages (Fig. 9):

$$[X_i[= c(a_{i_1},x',b_x,I_i) \bullet c(a_{i_2},x',x',x') \bullet \ldots \bullet c(a_{i_{r_i}},x',x',x'),$$

$$]X_i[= c(a_{i_1},x',x_i,x') \bullet c(a_{i_2},x',x',x') \bullet \ldots \bullet c(a_{i_{r_i}},x',x',x'),$$

$$]X_i] = c(a_{i_1},x',x',x') \bullet c(a_{i_2},x',x',x') \bullet \ldots \bullet c(a_{i_{r_i}},0,e_{x_i},x').$$

The sign $\bullet$ means that the colored squares are matched in this order on an horizontal line.

Thus, $[X_i[$ (resp. $[X_i[,]X_i]$) is a tile which will be used to encode the word x_i standing at the beginning (resp. the middle, the end) of a solution of the Post problem. In the same way we define $[Y_i[,]Y_i[,]Y_i]$, replacing the letter x by the letter y.

We get in this way $6k$ *basis-polyominoes*.

These polyominoes encode the words $x_1,\ldots,x_k, y_1,\ldots,y_k$ in three different ways. We define now *annex-polyominoes*: these polyominoes are used to convey information in some direction, up, down, right or left (Fig. 10).

For example $u.l$ means that the information is conveyed from up to left.

We have to add the annex-polyominoes obtained by changing x into y in Fig. 10. Letters B,E,M,N are chosen for *beginning*, *ending*, *middle*, *neutral*, and for example we will say that $BM_x[j,r.l]$ is of type BM. Let $\mathscr{C}$ be the set of $6k$ basis-polyominoes and $16k+2$ annex-polyominoes.

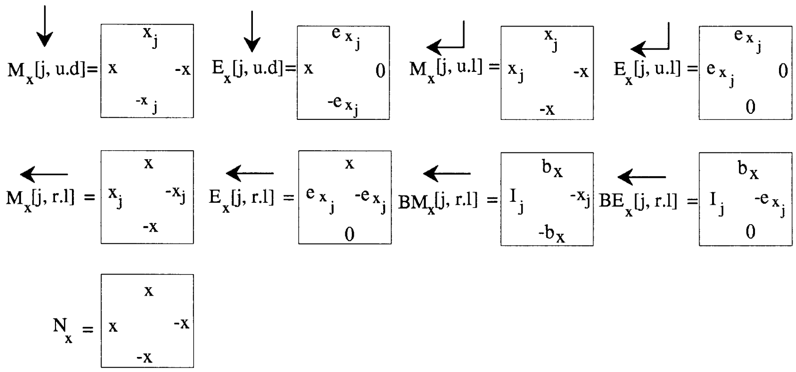

Fig. 10. Annex-polyominoes.

Proposition 1. *If the Post problem has a solution, then* $\mathscr{C}$ *is not a code.*

Proof. Let $(i_1,\ldots,i_n)$ be a finite sequence of integers belonging to $[1..k]$ such that:

$$x_{i_1}\ldots x_{i_n} = y_{i_1}\ldots y_{i_n}.$$

We put $|x_{i_1}\ldots x_{i_n}| = L$. Let P_x (resp. P_y) be the polyomino obtained joining horizontally from left to right the colored squares (this forces n to be at least equal to 2):

$[X_{i_1}[,]X_{i_2},[\ldots,]X_{i_{n-1}}[,]X_{i_n}]$ (resp. $[Y_{i_1}[,]Y_{i_2},[\ldots,]Y_{i_{n-1}}[,]Y_{i_n}]$).

Let p be the polyomino composed of the package of n rows of colored squares, each row being composed of the package of L colored squares such that:

- the first row of p is P_x,
- for $1<j<n$, the jth row of p is the package:

$$BM_x(i_j,r.l)M_x(i_j,r.l)^{|x_{i_1}\ldots x_{i_{j-1}}|-1}M_x(i_j,u.l)N_x^{|x_j|-1}M_x(i_{j+1},u.d)N_x^{|x_j+1|-1}/\ldots$$

$$\ldots/M_x(i_{j+2},u.d)\ldots M_x(i_{n-1},u.d)N_x^{|x_{i_{n-1}}x_{i_n}|-2}E_x(i_n,u.d),$$

- for $j=n$:

$$BE_x(i_n,r.l)E_x(i_n,r.l)^{|x_{i_1}\ldots x_{i_n}|-2}E_x(i_n,d.l).$$

So we obtain a polyomino p which is a “rectangle” of $L\times n$ colored squares such that its east and south sides are straight lines, on the north side we can “read” the word $a_{j_1}\ldots a_{j_L}$ and on the west side we “read” the integers $i_1,\ldots,i_n$ where $x_{i_1}\ldots x_{i_n} = a_{j_1}\ldots a_{j_L}$ (Fig. 11).

Clearly, p is also obtained from P_y in the same way, replacing x by y in each colored square used in the package, so it is proved that p admits two different tilings with $\mathscr{C}$. $\square$

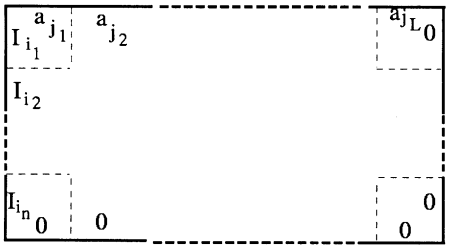

Fig. 11. The polyomino encoding the sequence $x_{i_1} \ldots x_{i_n}$.

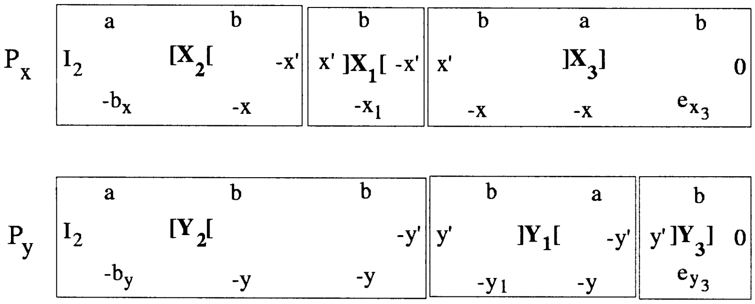

Fig. 12. The polyominoes P_x et P_y.

To illustrate this result, let us give an example:

$$A = \{a, b\}, \quad X = (x_1, x_2, x_3), \quad Y = (y_1, y_2, y_3),$$

$$x_1 = b, \quad x_2 = ab, \quad x_3 = bab, \quad y_1 = ba, \quad y_2 = abb, \quad y_3 = b.$$

We have $x_2x_1x_3 = y_2y_1y_3$.

The polyominoes P_x and P_y are (Fig. 12):

The polyomino p is obtained in two ways:

In Figs. 13 and 14, arrows show how the information about the sequence of indices $(j_1, \ldots, j_n)$ is conveyed by means of hooks from the highest row to the leftmost column of the rectangle.

Proposition 2. *If $\mathscr{C}$ is not a code, then the related Post problem has a solution.*

Proof. This reverse statement is a bit more complicated to be proved. Let p be a polyomino of minimal size (according to its number of cells) which admits at least two tilings with $\mathscr{C}$. Polyominoes of $\mathscr{C}$ are packages of colored squares of size $2m+1$. Thus, p is tiled with the set of colored squares with size $2m+1$. Let $G = (C, F)$ be the graph related to p as in Lemma 3. One can observe that, according to the definition of $\mathscr{C}$, if a colored square appearing in a basis-polyomino is used in a tiling of p, its center has no north neighbor, and so its north color is an invariant of p. Let us write

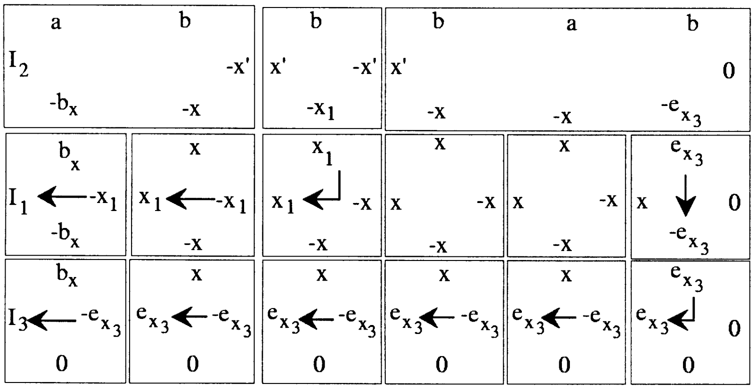

Fig. 13. A way to tile p.

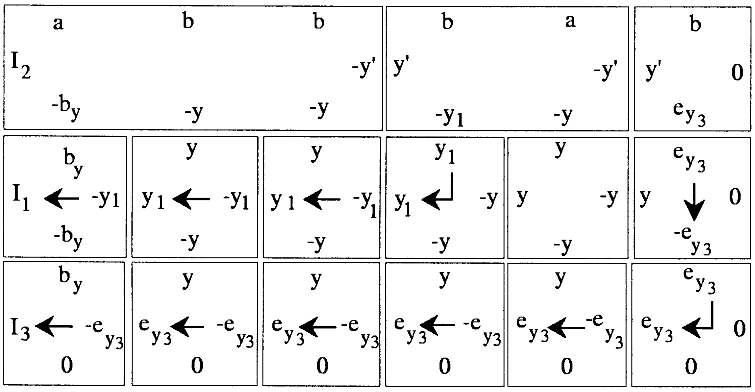

Fig. 14. Another way to tile p.

$C = C_1 \cup C_2$ with C_2 being the set of centers of annex-polyominoes, and C_1 being the set of centers of colored squares which basis-polyominoes are built of. It turns out that C_1 and C_2 are invariants of p. Let A be the highest point among the leftmost points of C. We can prove the following statement:

Claim 1. *If $A \in C_2$ then p is not of minimal size among the polyominoes admitting at least two tilings with $\mathscr{C}$.*

Proof of Claim 1. Let us denote by $c_T(A)$ the colored square with center A in the tiling T.

Actually, since $A \in C_2$, then A has no west nor north neighbor, so (Lemma 2) $col_W(c_T(A))$ and $col_N(c_T(A))$ are invariants of p, that is they do not depend on T.

Two cases arise:

Case 1: $c_T(A)$ is not of type BM or BE.

Then $c_T(A)$ is entirely defined by its west and north colors. For example if $col_W(c_T(A)) = x_j$ and $col_N(c_T(A)) = x$, then $c_T(A) = M_x[j, r.l]$.

In this case, $c_T(A)$ is an invariant of p whence p is not minimal because we can remove the N–W maximal colored square of p which is the same in all the tilings of p with $\mathscr{C}$.

Case 2: $c_T(A)$ is of type BM or BE.

In this case, let A' be the end of the maximal path in G beginning in A and going down. The point A' has no south neighbor in G, so (Lemma 2) $col_S(c_T(A'))$ is an invariant of p. There are only two possibilities:

- if $A = A'$ then $col_S(c_T(A))$ is an invariant of p and then $c_T(A)$ is also an invariant of p whence p is not minimal as above.
- if $A \neq A'$ then $c_T(A)$ is some $BM_z[i, r.l]$ or some $BE_z[i, n.l]$ for some i and $z \in \{x, y\}$, and p is nor minimal.

End of the proof of Claim 1. Hence, using Claim 1, necessarily A belongs to C_1.

Let $(A_1, \ldots, A_L)$ be the maximal path in G beginning in $A = A_1$ going on the right. For $i \in \{1, \ldots, L\}$, $A_i \in C_1$ because the color x' (or y') forces the A_i's to be in C_1. So A_i has no north neighbor. Then, $col_N(c_T(A_i))$ is an invariant of p for $i \in \{1, \ldots, L\}$. On account of the maximality of the path, $col_E(c(A_L))$ is an invariant of p.

Claim 2. *If* $col_W(c_T(A_1)) \notin \{I_i \mid i \in \{1, \ldots, k\}\}$ *or* $col_E(c_T(A_L)) \neq 0$ *then* p *is not of minimal size.*

Proof of Claim 2.

Case 1: $col_E(c_T(A_L)) \neq 0$ and $col_W(c_T(A_1)) \notin \{I_i \mid i \in \{1, \ldots, k\}\}$.

One can observe that all the east and west colors of $A_1, \ldots, A_L$ have the same absolute value and belong to $\{x', y', -x', -y'\}$. Therefore, $c_T(A_1)$ is the first left square of some $]X_i[$ or some $]Y_i[$. Let us consider the point A' defined as before.

- if $A = A'$, then $col_S(c_T(A_1))$ is an invariant of p and belongs to the set $\{-x_i, -y_i \mid i \in \{1, \ldots, k\}\}$. Thus, if $col_W(c_T(A_1)) = x'$ (resp. y'), and $col_S(c_T(A_1)) = -x_i$ (resp. $-y_i$) then $c_T(A)$ is an invariant of p, and every tiling uses polyomino $]X_i[$ (resp. $]Y_i[$) in such a way that its first left square is centered in A. Here again, p is not minimal.
- if $A \neq A'$ then:
  - if $col_S(c_T(A')) = 0$ then $col_S(c_T(A')) = col_S(c_T(A))$ and the same argument holds.
  - if $col_S(c_T(A')) = 0$, A' is of type M or N and its west and south colors are invariants of p.

Three possibilities: the (west, south) color of $c_T(A')$ are $(x, -x_j)$ or $(x, -x)$, or $(x_j, -x)$ for some j (we reason with x but the reasoning is similar for y).

If the (west, south) color of $c_T(A')$ are $(x, -x_j)$ then $c_T(A')$ is equal to $M_x[j, u.d]$ and is an invariant of p.

If the (west, south) color of $c_T(A')$ are $(x, -x)$ then $c_T(A')$ is equal to N_x and is an invariant of p.

If the (west, south) color of $c_T(A')$ is $(x_j, -x)$ then $c_T(A')$ is equal to $M_x[j, r.l]$ or $M_x[j, u.l]$. But in this last case, there is on the path from A (exclude A) to A' a first point A'' such that $col_W(c_T(A''))=x_j$ for some j. And then, $c_T(A'')$ is necessarily equal to $M_x[j,u.l]$ and is an invariant of p.

Thus if $A\neq A'$ then p is not minimal.

Case 2: $col_E(c_T(A_L))=0$ and $col_W(c_T(A_1))\notin\{I_i\mid i\in\{1,\dots,k\}\}$.

Then $col_W(c_T(A_1))\in\{x',y'\}$ and it is an invariant of p. Let B be the end of the maximal path in G beginning in A_L and going down; $col_S(c_T(B))$ is an invariant of p.

- if $B=A_L$ then $col_S(c_T(A_L))$ belongs to the set $\{-e_{x_i},e_{y_i}\mid i\in\{1,\dots,k\}\}$. Thus, if $col_W(c_T(A_1))=x'$ (resp. y') and $col_S(c_T(A_L))=-e_{x_i}$ (resp. $-e_{y_i}$), then every tiling of p uses polyomino $]X_i]$ (resp. $]Y_i]$) in such a way that its last right square is centered in A_L. Hence p is not with minimal size.
- if $B\neq A_L$, two cases are possible:

(1) if $col_S(c_T(B))\in\{e_{x_i},e_{y_i}\mid i\in\{1,\dots,k\}\}$ then $col_S(c_T(A_L))=col_S(c_T(B))=e_{x_i}$ (resp. e_{y_i}) for some i and the above argument can be repeated: every tiling of p uses polyomino $]X_i]$ (resp. $]Y_i]$) in such a way that its last right square is centered in A_L. Hence p is not with minimal size.

(2) if $col_S(c_T(B))=0$, let B' be the beginning of the maximal horizontal path in G which halts in B; $col_W(c_T(B'))$ is an invariant of p.

(2.a) if $col_W(c_T(B'))\in\{e_{x_i},e_{y_i}\mid i\in\{1,\dots,k\}\}$, we conclude in the same way as above (because $col_W(c_T(B'))=col_S(c_T(A_L))$).

(2.b) if $col_W(c_T(B'))=I_i$, recall that $col_W(c_T(A_1))=x'$ (resp. y'). So every tiling of p uses some polyomino $]X_i]$ (resp. $]Y_i]$) in such a way that its last right square is centered in A_L. Hence p is not of minimal size.

Case 3: $col_E(c_T(A_L))\neq 0$ and $col_W(c_T(A_1))\in\{I_i\mid i\in\{1,\dots,k\}\}$.

In this case, $col_E(c_T(A_L))=-x'$ (resp. $-y'$). Then, every tiling of p uses some polyomino $[X_i[$ (resp. $[Y_i[$) in such a way that its last right square is centered in $A_1=A$. Hence p is not of minimal size.

End of the proof of Claim 2. As a consequence of Claim 2, we have obtained the following conclusion: every maximal horizontal path $(A=A_1,\dots,A_L)$ has the following property:

$col_W(c_T(A_1))\in\{I_i\mid i\in\{1,\dots,k\}\}$, $col_E(c_T(A_L))=0$, and for $1\leqslant j\leqslant L$, $A_j\in C_1$.

Let $(A'_1,\dots,A'_r)$ be the maximal path in G beginning in $A'_1=A$ and going down. Using minimality of p, we conclude that $r\neq 1$ and $A'_r\in\{BE_x(i,r.l),BE_y(i,r.l)\mid i\in\{1,\dots,k\}\}$. Let $I_{m_l}=col_W(c_T(A'_l))$ for $l\in\{1,\dots,r\}$; $c_T(A_1)\dots c_T(A_L)$ is built with a package $[X_{i_1}[\dots]X_{i_{r'}}]$ (or $[Y_{n_1}[\dots]Y_{n_{r''}}]$). Let us suppose that

$$c_T(A_1)\dots c_T(A_L)=[X_{i_1}[\dots]X_{i_{r'}}].$$

Then we have $x_{i_1}\dots x_{i_{r'}}=col_N(c(A_1))\dots col_N(c(A_L))$. Now we just prove two statements:

- *$r=r'$ and for every $l\in\{1,\dots,r\}$ $m_l=i_l$,*
- *p contains the rectangular polyomino P_x we built in Prop.* 1, *related to the sequence $x_{i_1}\dots x_{i_r}$ in which the leftmost square is centered in A.*

First of all, $col_W(c_T(A_1))=I_{m_1}$ and $m_1=i_1$. Secondly, it is obvious that if $m_2\neq i_2$ then the maximal path in G starting in A'_2 and going on the right has a length strictly less than L and in that case, p is not of minimal size because the east color of the extremity of this path is an invariant of p which allows us to find what kind of polyomino is on A. So, $m_2 = i_2$, and using minimality of p, the maximal path in G starting in A'_2 has length L and the second row has the following structure:

if $r' > 2$:

$$BM_x(i_2, r.l)M_x(i_2, r.l)^{|x_{i_1}|-1}M_x(i_2, u.l)N_x^{|x_{i_2}|-1} \\ \dots M_x(i_{r-1}, u.d)N_x^{|x_{i_{r-1}}x_{i_r}|-2}E_x(i_r, u.d),$$

if $r' = 2$:

$$BE_x(i_2, l.r)E_x(i_2, r.l)^{|x_{i_1}x_{i_2}|-2}E_x(i_2, u.l).$$

Recurrently, the related property is proved for the other rows up to r. Thus, if only one of the two words $x_{i_1} \dots x_{i_r}$, $y_{i_1} \dots y_{i_r}$ is equal to $col_N(c(A_1)) \dots col_N(c(A_L))$ then p is not of minimal size. Therefore, we get $x_{i_1} \dots x_{i_r} = y_{i_1} \dots y_{i_r}$ and it is exactly what we had to prove. □

Theorem 1. *One cannot decide whether a finite set of polyominoes is a code.*

Proof. It is a direct consequence of Propositions 1 and 2. □

This undecidability result holds also for dominoes.

Let $Col = \{-m, \dots, 0, \dots, m\}$ be a set of *colors*. A *domino* is a 1×1 square, fixed in orientation, whose edges are "colored" with elements of Col. A finite set of domino types T is given. A *colored tile* is a finite set of adjacent dominoes without hole, such that two adjacent dominoes have opposite colors on their common edge. A finite set $\mathscr{C}_d$ of dominoes is a code if every colored tile covered by nonoverlapping translated copies of tiles of $\mathscr{C}_d$ respecting the colors rule is covered in a unique way. Let us state the "codicity problem" for dominoes:

Theorem 2. *The problem whether a finite set of dominoes is a code is undecidable.*

Proof. We give here just a sketch of the proof. First of all, let $\mathscr{C}$ be a subset of colored squares in $\mathscr{C}_m$. Consider the set of dominoes $\mathscr{C}_d$ built from $\mathscr{C}$ by replacing each colored square c by a domino c' with color $|i|$ on a side iff c has color i or $-i$ on this same side. Clearly $\mathscr{C}$ is a code iff $\mathscr{C}_d$ is a code. The encoding of the Post problem by a set $\mathscr{C}'$ of colored squares (instead of a set of polyominoes) is a little bit more complicated. If we start with the previous encoding, by a set $\mathscr{C}$ of polyominoes, polyominoes $[X_i[,]X_i[,]X_i]$ (resp. $[Y_i[,]Y_i[,]Y_i]$) are to be "cut" in as many colored squares as necessary. In order to do this in a proper way, we are to introduce at each vertical "cut" a new color (and its opposite) which forces the colored squares to match only for rebuilding the original polyominoes. As a consequence, the number of colors being greater, the size m has to be increased. Let $\mathscr{C}'$ be the set of colored squares we get in this way. We claim that the Post problem has a solution iff $\mathscr{C}'$ is not a code. Using Proposition 1, if the Post problem has a solution, $\mathscr{C}$ is not a code and thus neither $\mathscr{C}'$. For the other direction, the proof follows the same steps as in Proposition 2, but the proof of Claim 2 is a little bit more intricated. □

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